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A note on the average shadowing property for expansive maps

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ABSTRACT

Let f be a continuous map of a compact metric space. Assuming shadowing for f we relate the average shadowing property of f to transitivity and its variants. Our results extend and complete the work of Sakai [K. Sakai, Various shadowing properties for positively expansive maps, Topology Appl. 131 (2003) 15–31].

1. Introduction

The notion of average pseudo-orbit was introduced by Michael Blank in [2] in order to investigate pseudo-orbits arising naturally as realizations of dynamical system stochastically perturbed by independent Gaussian random variables with zero mean (see also [3] and [4, Section 6.3]). The notion of average shadowing property was further studied by several authors, with particular emphasis on connections with other notions known from topological dynamics, or more narrowly, shadowing theory (e.g. see [13]). The reader not familiar with various shadowing and expansivity properties can find their definitions in the next section.

In his paper [13], Sakai considered conditions sufficient for the average shadowing property (as well as other kinds of shadowing properties) in the setting of positively expansive maps. The present work aims to extend and complete some investigations of [13].

The main result of this short paper is:

Theorem 1. *Let X be a compact metric space. If $f : X \to X$ is a continuous map with the shadowing property, then the following conditions are equivalent*:

(1) *f is totally transitive,*
(2) *f is topologically weakly mixing,*

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(3) *f is topologically mixing,*
(4) *f is surjective and has the specification property,*
(5) *f is surjective and has the average shadowing property.*

Moreover, if f is in addition a c-expansive map, then any of the above conditions is equivalent to the periodic specification property of f.

As a direct consequence of our Theorem 1 and [13, Theorem 1] we obtain:

Corollary 2. *Let (X,d) be a compact metric space. If $f:X\to X$ is a positively expansive and open continuous map, then the following conditions are equivalent*:

(1) *f is totally transitive,*
(2) *f has the average shadowing property.*

Proof. By [13, Theorem 1], f has the shadowing property and then Theorem 1 applies, since every positively expansive maps is c-expansive (see [1]). □

Our corollary is a corrected version of [13, Theorem 2]. More precisely, [13] contains (among the others valuable results) a proof of the following:

Statement 1 *([13, Theorem 2]).* Let $f:X\to X$ be a positively expansive open map acting on a compact metric space. The following conditions are equivalent*:*

(1) f has average shadowing property,
(2) f is transitive.

Unfortunately, the implication (2) $\Rightarrow$ (1) is not true under given assumptions, as there are many examples of transitive positively expansive and open systems without the average shadowing property. One can prove that every one-sided shift of finite type, which is transitive, but not totally transitive provides a counterexample for that implication. It is well known that all shifts of finite type are open and positively expansive. One way to see that they do not posses the average shadowing property it is to repeat the reasoning presented in Example 3 below. It also follows from our Corollary 2. A careful analysis of [13] reveals that although subtle idea of the proof of Theorem 2 in [13] is correct, one of auxiliary facts used there, namely [13, Lemma 3] does not hold under the assumption that f is merely transitive. As we believe that [13, Lemma 3] has the value by itself and it is a useful tool in analysis of dynamics of positively expansive maps, we conclude our work with the proof of a version of it under stronger assumption of weak mixing. Substituting [13, Lemma 3] by our Theorem 14 and repeating the rest of reasoning presented by Sakai one obtains a valid proof of Corollary 2, which does not refer directly to the notion of specification.

The rest of the paper is organized as follows: in the next section we provide definitions and introduce notation for the rest of the paper. Section 3 contains the details of our counterexample for [13, Theorem 2]. In Section 4 we prove our main theorem, and finally in Section 5 we prove Theorem 14. As our work contains remarks to the work of some other mathematician, we wanted it to be more self-contained than a usual research paper. Hence, in a few lemmas we decided to include direct proofs of results which can be obtained indirectly by a reference to the literature.

2. Preliminaries on topological dynamics

Let $f:X\to X$ be a continuous map acting on a compact metric space (X,d). Throughout this paper k, l, m, and n always denote integers. An *orbit* of a point $x\in X$ is the set $\{f^n(x):\ n\geqslant 0\}$. A point $x\in X$ is *periodic* if there is $n>0$ such that $f^n(x)=x$, and any positive n such that $f^n(x)=x$ is called a *period* of x. The *least period* of a periodic point x is the smallest period. A *full orbit* through $x\in X$ is a bi-infinite sequence $\{x_k\}_{k\in\mathbb{Z}}$ such that $x_0=x$ and $f(x_k)=x_{k+1}$ for every $k\in\mathbb{Z}$. If f is a surjection then through every point $x\in X$ there is at least one full orbit.

We say that f is *transitive* if for every pair of nonempty open subsets U and V of X there is an $n\geqslant 0$ such that $f^n(U)\cap V\neq\emptyset$; f is *totally transitive* if all its iterates f^n are transitive; f is *weakly mixing* if $f\times f$ is transitive, where $f\times f:X\times X\to X\times X$ is a continuous map given by $f\times f(x,y)=(f(x),f(y))$; and finally, f is *mixing* if for any nonempty sets U and V open in X, there is an $N>0$ such that $f^n(U)\cap V\neq\emptyset$ for all $n\geqslant N$. We say that f has *dense small periodic sets* if for every nonempty open set $U\subset X$ there is a closed subset $A\subset U$ and $n>0$ such that $f^n(A)=A$. The later property was introduced by Huang and Ye [8] and it is closely related to Smale's spectral decomposition (see [10], where it appears as property (P)).

The specification property was introduced by Bowen in [6] (see also [7]). We say that f has the *periodic specification property* if, for any $\varepsilon>0$, there is an integer $N_\varepsilon>0$ such that for any integer $s\geqslant 2$, any set $\{y_1,\dots,y_s\}$ of s points of X, and any sequence $0 = j_1 \leqslant k_1 < j_2 \leqslant k_2 < \cdots < j_s \leqslant k_s$ of $2s$ integers with $j_{l+1} - k_l \geqslant N_\varepsilon$ for $l = 1, \dots, s-1$, there is a point $x \in X$ such that, for each $1 \leqslant m \leqslant s$ and any i with $j_m \leqslant i \leqslant k_m$, the following conditions hold:

$$d\big(f^i(x), f^i(y_m)\big) < \varepsilon, \tag{1}$$

$$f^n(x) = x, \quad \text{where } n = N_\varepsilon + k_s. \tag{2}$$

If we drop the periodicity condition (2) from the above definition, that is, if f fulfills only the first condition above, then we say that f has the *specification property*.

Neither maps with the specification property, nor maps with the (average) shadowing property have to be onto. To see it, consider $X = \{a, b\}$, and $f : X \mapsto X$ given by $f(a) = f(b) = a$. It is not hard to verify that the map f defined above has the specification property, and shadowing property as well as average shadowing property. Therefore, we need to assume separately that f is surjective in Theorem 1, and in a few results to follow.

Now, let us recall three main definitions for this paper. We say that f is *c-expansive* if there is a constant $c > 0$ (called an *expansive constant*) such that if $x, y \in X$, $\{x_k\}_{k\in\mathbb{Z}}$, $\{y_k\}_{k\in\mathbb{Z}}$ are full orbits through x and y, respectively, and the condition $d(x_k, y_k) < c$ holds for all $n \in \mathbb{Z}$, then $x = y$. A map f is *positively expansive* if there is a constant $e > 0$ such that if $x, y \in X$ and $d(f^n(x), f^n(y)) < e$ hold for all $n \geqslant 0$, then $x = y$. Any positively expansive map is c-expansive, but not conversely.

We recall that a sequence of points $\{x_n\}_{n=0}^\infty$ is called a *δ-pseudo-orbit* if $d(f(x_n), x_{n+1}) < \delta$ for $n = 0, 1, 2, \dots$. Given $\varepsilon > 0$ and $\delta > 0$, we say that a δ-pseudo-orbit $\{x_n\}_{n=0}^\infty$ is *ε-traced* by a point $y \in X$ when $d(f^n(y), x_n) < \varepsilon$ for $n = 0, 1, \dots$. We say that f has the *pseudo-orbit tracing property* (*shadowing* for short) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit in X is ε-traced by some point in X.

We say that a sequence $\{x_n\}_{n=0}^\infty$ is a δ-average-pseudo-orbit, if there is an integer $N_\delta > 0$ such that for all $n \geqslant N_\delta$ and $k \geqslant 0$ the following condition is satisfied

$$\frac{1}{n}\sum_{i=0}^{n-1} d\big(f(x_{i+k}), x_{i+k+1}\big) < \delta.$$

We say that f has the *average shadowing property* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that every δ-average-pseudo-orbit is *shadowed on average* by some point $y \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d\big(f^i(y), x_i\big) < \varepsilon.$$

Two metrics d, and ρ on X are said to be *uniformly equivalent* if the identity map $\mathrm{id} : (X, d) \to (X, \rho)$ is uniformly continuous with a uniformly continuous inverse. It is well known that two metrics on a given compact metrizable topological space X are uniformly equivalent if and only if they are equivalent, that is, they are both compatible with the topology of X.

The definitions of shadowing property, average shadowing property, positive expansivity, and c-expansivity given above refers to the metric on X, but in fact they depend only on topology of a compact metric space X, meaning that if $f : X \to X$ has one of the above properties with respect to some metric d on X generating the topology of X, then f has that property with respect to every metric ρ on X equivalent, hence uniformly equivalent with d.

3. The example

The aim of this section is to show by example that the implication (2) ⇒ (1) in [13, Theorem 2] is false. We present here the simplest example available, but the same reasoning would also work in other situations.

Example 3. Let $X = \{a, b\}$ be any two points set with the discrete metric d and let f be the cyclic permutation of X, that is, $f(a) = b$, $f(b) = a$. Then f is open, positively expansive and transitive, but it does not have the average shadowing property.

Proof. It is obvious that f is open, positively expansive and transitive. We will prove that it does not have the average shadowing property.

Fix $\varepsilon = 1/4$ and take any $\delta > 0$ and let m be such that $1/m < \delta$. Put $N = 2m$ and consider the sequence

$$x_i = \begin{cases} f^j(a), & \text{if } i = 2sN + j \text{ for some } s \geqslant 0, \text{ and } 0 \leqslant j < N, \\ f^j(b), & \text{if } i = (2s+1)N + j \text{ for some } s \geqslant 0, \text{ and } 0 \leqslant j < N. \end{cases}$$

In other words,

$$x_0, x_1, x_2, \dots = \underbrace{a, b, a, b, \dots, a, b}_{2m}, \underbrace{b, a, b, a, \dots, b, a}_{2m}, \underbrace{a, b, \dots, a, b}_{2m}, \dots$$

We are going to show that the sequence $\{x_i\}_{i=0}^{\infty}$ is a δ-average-pseudo-orbit, with $N_\delta = N$ given above. Let $n \geqslant N$ be given, say $n = lN + j$ for some $l \geqslant 1$, and $0 \leqslant j < N$. If we fix any $k \geqslant 0$, then the set $\{i\colon\ 0 \leqslant i < n \text{ and } d(f(x_{i+k}), x_{i+k+1}) \neq 0\}$ has at most $l + 1$ elements. Since $d(f(x_i), x_{i+1}) \neq 0$ implies $d(f(x_i), x_{i+1}) = 1$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} d\bigl(f(x_{i+k}), x_{i+k+1}\bigr) \leqslant \frac{1}{lN}(l+1) \leqslant \frac{2}{N} = \frac{1}{m} < \delta$$

which shows that the sequence $\{x_i\}_{i=0}^{\infty}$ is indeed a δ-average-pseudo-orbit. For any $s \geqslant 1$ the following holds

$$\begin{aligned}\frac{1}{2Ns}\sum_{i=0}^{2Ns-1} d\bigl(f^i(a), x_i\bigr) &= \frac{1}{2Ns}\sum_{i=0}^{s-1}\sum_{j=0}^{N-1} d\bigl(f^{N(2i+1)+j}(a), f^j(b)\bigr)\\ &= \frac{1}{2Ns}\sum_{i=0}^{s-1}\sum_{j=0}^{N-1} d\bigl(f^j(a), f^j(b)\bigr) = \frac{1}{2} > \varepsilon = \frac{1}{4},\end{aligned}$$

which implies

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d\bigl(f^i(a), x_i\bigr) \geqslant \varepsilon.$$

By the same arguments we also get

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d\bigl(f^i(b), x_i\bigr) \geqslant \varepsilon.$$

In other words, for $\varepsilon = 1/4$, the δ-average-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ constructed above cannot be ε-shadowed in average by any point of the space X, which proves the claim. $\square$

As we mentioned in the introduction, Example 3 is the simplest possible. But the same will hold if we consider any transitive but not mixing one-sided shift of finite type. The proof is an easy modification of the above reasoning. It raises the following problem:

Question 1. Is there any transitive, but not totally transitive system with the average shadowing property?

4. Proof of Theorem 1

The proof of Theorem 1 will be divided into a sequence of lemmas, since some of them may be interesting in their own right.

Throughout this section if (X, d) is a compact metric space, $x \in X$ and $r > 0$, then $B(x, r)$ denotes an open ball centered at x, and with the radius r.

The proof of the following lemma is straightforward, therefore we omit it. Note that, on $X \times X$ we consider the max metric $d_{\max}$ defined by $d_{\max}((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), d(x_2, y_2)\}$.

Lemma 4. *Let $f : X \to X$ be a continuous map of a compact metric space (X, d).*

(1) *If f is c-expansive, then so is $f \times f$.*
(2) *If f has the shadowing property, then the same holds for $f \times f$.*
(3) *If f has the (periodic) specification property, then the same holds for $f \times f$.*

We need the following lemma which is a special case ($X = Y$, and $f = g$) of [9, Proposition 3.5]. Its proof is straightforward.

Lemma 5. *If $f : X \to X$ is a continuous map of a compact metric space (X, d) with the average shadowing property, then the same holds for $f \times f$.*

Next lemma appeared as Proposition 3.7 in [11] and is simple to prove, but again we present it for completeness.

Lemma 6. *If $f : X \to X$ is a totally transitive continuous map of a compact metric space (X, d) with dense small periodic sets, then f is weakly mixing.*

Proof. Let U, V, Y, Z be nonempty open subsets of X. Since f is transitive, there is an $n > 0$ such that $W = f^{-n}(Y) \cap U$ is a nonempty open set. Since f has dense small periodic sets, there is a nonempty closed set $A \subset W$ and $k > 0$ such that $f^k(A) = A$. Hence, $f^{n+jk}(U) \cap Y \neq \emptyset$ for each $j > 0$. Since f^k is still transitive there is a $j_0 > 0$ such that $f^{j_0 k}(V) \cap f^{-n}(Z) \neq \emptyset$. Therefore, $(f \times f)^{n+j_0 k}(U \times V) \cap (Y \times Z) \neq \emptyset$, which implies that f is weakly mixing. $\square$

Lemma 7. *If $f : X \to X$ is a transitive continuous map of a compact metric space (X, d) with the shadowing property, then f has dense small periodic sets.*

Proof. Let $U \neq \emptyset$ be an open subset of X. Let us choose $\varepsilon > 0$ such that $V = B(u, 3\varepsilon) \subset U$ for some $u \in U$. For that $\varepsilon > 0$ we take $\delta > 0$ as in the definition of the shadowing property. Without loss of generality we assume that $\delta < \varepsilon$. By transitivity of f there is an $n > 0$ and $x \in B(u, \varepsilon)$ such that $d(f^n(x), x) < \delta$. Therefore the sequence $\{y_k\}_{k=0}^{\infty}$ given by $y_k = f^{(k \bmod n)}(x)$ is a periodic δ-pseudo-orbit for f. By shadowing there is a point $z \in B(x, \varepsilon) \subset B(u, 2\varepsilon)$ such that for all $k \geqslant 0$ we have $d(f^{nk}(z), x) < \varepsilon$. Therefore, $\omega(z, f^n) \subset \bar{B}(u, 3\varepsilon)$, and setting $A = \omega(z, f^n)$, we see that $f^n(A) = A \subset U$, hence f has dense small periodic sets. $\square$

Lemma 8. *If $f : X \to X$ is a weakly mixing continuous map of a compact metric space (X, d) with shadowing, then f is mixing.*

Proof. Let U, V be nonempty open subsets of X. Let $\varepsilon > 0$ be such that $U' = B(u, 2\varepsilon) \subset U$ and $V' = B(v, 2\varepsilon) \subset V$ for some $u \in U$ and $v \in V$. For that $\varepsilon > 0$ we take $\delta > 0$ as in the definition of the shadowing property. Without loss of generality we assume $\delta < \varepsilon$. By Lemma 7 f has dense small periodic sets. Therefore there is an $n > 0$, and a nonempty closed set $A \subset U'' = B(u, \delta/2)$ such that $f^n(A) = A$. We fix $a \in A$ and note that the sequence $\{a_k\}_{k=0}^{\infty}$ given by $a_k = f^{(k \bmod n)}(a)$ is a periodic δ-pseudo-orbit for f. By weak mixing, there is $m > 0$ such that for each $j = 0, \dots, n-1$ there is a point $z_j \in U''$ such that $f^{m+j}(z_j) \in V'$. Now for each $l \geqslant m$ where $l = m + sn + j$, $s \geqslant 0$, and $0 \leqslant j \leqslant n-1$ we define a sequence $\{y_t^l\}_{t=0}^{\infty}$ by

$$y_t^l = \begin{cases} a_t, & \text{for } t = 0, \dots, sn-1, \\ f^{t-sn}(z_j), & \text{for } t \geqslant sn. \end{cases}$$

Since, $f^{ns}(a)$, $z_j \in B(u, \delta/2)$, we have $d(f^{ns}(a), z_j) < \delta$, so $\{y_t^l\}_{t=0}^{\infty}$ is a δ-pseudo-orbit for f such that $y_l^l \in V'$. Assume that the orbit of a point x_l ε-traces $\{y_t^l\}_{t=0}^{\infty}$. Then $f^l(x_l) \in V$, therefore $f^l(U) \cap V \neq \emptyset$ for each $l \geqslant m$. $\square$

Alternatively, the two lemmas above can be derived from [12, Corollary 12], where it is proved that a continuous map of a compact metric space is totally chain transitive if and only if it is chain mixing. As every totally transitive map must be totally chain transitive, and any chain mixing map with shadowing must be topologically mixing, the implication

$$f \text{ totally transitive} \quad \Longrightarrow \quad f \text{ mixing}$$

holds, provided f has the shadowing property.

The following lemma was implicitly contained in [5] (see also [7, Proposition 23.20]).

Lemma 9. *Let (X, d) be a compact metric space. Assume that $f : X \mapsto X$ is a continuous mixing map with the shadowing property. Then f is surjective and has the specification property. If, in addition, f is c-expansive, then f has the periodic specification property.*

Proof. The proof that mixing map on compact space must be surjective is straightforward. To prove that f has the specification we fix $\varepsilon > 0$. For $\varepsilon/2$ let δ be as in the shadowing property of f. Clearly, we may assume $\delta < \varepsilon/2$. By compactness of X we can find a finite open cover $U_1, \dots, U_M$ of X such that $\operatorname{diam}(U_i) < \delta$ for every i. Since f is mixing, we can find an integer $N > 0$ such that for any $i, j \in \{1, \dots, M\}$ and $L \geqslant N$ there is a point $x_{i,j}^L \in U_i$ such that $f^L(x_{i,j}^L) \in U_j$. We claim that it is enough to set $N_\varepsilon = N$. We fix any integer $s \geqslant 2$, a set $\{y_1, \dots, y_s\} \subset X$, and a sequence $0 = j_1 \leqslant k_1 < j_2 \leqslant k_2 < \dots < j_s \leqslant k_s$ of $2s$ integers with $j_{l+1} - k_l \geqslant N_\varepsilon$ for $l = 1, \dots, s-1$. For $1 \leqslant m \leqslant s$ we let $b(m)$ and $t(m)$ to be such integers that $f^{j_m}(y_m) \in U_{b(m)}$ and $f^{k_m}(y_m) \in U_{t(m)}$, and put

$$y'_m = x_{t(m),b(m+1)}^{j_{m+1}-k_m}$$

(here, we agree that $b(s+1) = b(1)$ and $j_{s+1} = k_s + N$). We will use the shadowing property to define a point $x \in X$ such that the condition (1) in the definition of the specification is fulfilled.

First, we define a finite δ-pseudo-orbit by setting $z_i = f^i(y_m)$ for each $1 \leqslant m \leqslant s$ and any i with $j_m \leqslant i < k_m$, and $z_i = f^{i-k_m}(y'_m)$ for each $1 \leqslant m \leqslant s$ and any i with $k_m \leqslant i < j_{m+1}$, where we again agree that $j_{s+1} = k_s + N$. Then we can extend $\{z_i\}_{i=0}^{N+k_s-1}$ to the periodic bi-infinite δ-pseudo-orbit and find a point $x \in X$ which $\varepsilon/2$-traces it. It is easy to see that the condition (1) holds.

If we assume additionally that f is c-expansive then without loss of generality we may assume that ε above is smaller than expansivity constant for f. It is now straightforward to see that a periodic bi-infinite δ-pseudo-orbit can only be $\varepsilon/2$-traced by the unique periodic point, that is, $f^{N+k_s}(x) = x$. $\square$

According to the referee the following lemma can be proved along the same lines as a result from [14] stating that the average shadowing property implies chain-transitivity for homeomorphism of compact spaces. We include a new proof for completeness.

Lemma 10. *Let (X,d) be a compact metric space. If $f: X \mapsto X$ is a continuous surjection with the average shadowing and shadowing properties, then f is transitive.*

Proof. Without loss of generality we may assume that $\operatorname{diam} X \leqslant 1$. Fix any nonempty open sets $U, V \subset X$ and choose points $p \in U$, and $q \in V$. Let $\varepsilon > 0$ be such that $B(p,\varepsilon) \subset U$ and $B(q,\varepsilon) \subset V$. For that ε take $\delta > 0$ provided by the shadowing property. By uniform continuity we can find $\eta < \delta$ such that $d(x,y) < \eta$ implies $d(f(x), f(y)) < \delta$ for any $x, y \in X$. For $\eta/3$ we take $\xi > 0$ as in the definition of average shadowing property.

Let N be an integer such that $2/\xi < N$ and let $r \in X$ be such that $f^N(r) = q$. Consider the sequence

$$x_i = \begin{cases} f^j(p), & \text{if } i = 2s \cdot N + j \text{ for some } s \geqslant 0 \text{ and } 0 \leqslant j < N, \\ f^j(r), & \text{if } i = (2s+1) \cdot N + j \text{ for some } s \geqslant 0 \text{ and } 0 \leqslant j < N. \end{cases}$$

In other words,

$$x_0, x_1, x_2, \ldots, x_{2N-1} = \underbrace{p, f(p), \ldots, f^{N-1}(p)}_{N} \underbrace{r, f(r), \ldots, f^{N-1}(r)}_{N}$$

and this initial sequence repeats *ad infinitum*. Analogously as in Example 3 one can easily show that $\{x_i\}_{i=0}^{\infty}$ constructed above is a periodic ξ-average-pseudo-orbit. By the average shadowing property we can find a point $z \in X$, which $\eta/3$-traces on average our ξ-average-pseudo-orbit.

We claim that for any $K > 0$ there is an integer σ such that $2\sigma N > K$ and $d(f^j(z), f^l(p)) < \eta$ for some $j \in [2\sigma N, (2\sigma + 1)N)$ and $l = i - 2\sigma N$. Assume that our claim does not hold, that is, for some σ_0 and all $s \geqslant \sigma_0$ we have $d(f^i(z), f^l(p)) \geqslant \eta$ for all $i \in [2sN, (2s+1)N)$ and $l = i - 2sN$. It follows that for all $s \geqslant \sigma_0$ we have

$$\sum_{i=2sN}^{(2s+1)N-1} d\big(f^i(z), x_i\big) \geqslant N\eta,$$

which implies

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d\big(f^i(z), x_i\big) \geqslant \eta/2 > \eta/3,$$

contradicting the fact that $z \in X$ is $\eta/3$-tracing on average our ξ-pseudo-orbit $\{x_i\}$. By our claim, there are $i > 0$ and $0 \leqslant l < N$ such that $d(f^i(z), f^l(p)) < \eta$. In the same way we prove that there are integers j, and m, where $j > i$, and $0 \leqslant m < N$ such that $d(f^j(z), f^m(r)) < \eta$.

By the choice of η, and i, j, l, m, the sequence

$$p, f(p), \ldots, f^l(p), f^{i+1}(z), f^{i+2}(z), \ldots, f^j(z), f^{m+1}(r), \ldots, f^N(r), f^{N+1}(r), \ldots$$

is a δ-pseudo-orbit of f, and thus by shadowing there is a point y which ε-traces it. We see that $y \in B(p,\varepsilon) \subset U$, and $f^{j-i+l+N-m-2}(y) \in B(f^N(r), \varepsilon) \subset V$, so indeed f is transitive. $\square$

Lemma 11. *Let (X,d) be a compact metric space. If $f: X \mapsto X$ is a continuous surjection with the average shadowing and shadowing properties, then f is weakly mixing.*

Proof. Observe that $f \times f$ also has the shadowing property and the average shadowing property, by Lemmas 4 and 5, respectively. Thus $f \times f$ is transitive by Lemma 10. $\square$

Alternatively, the two lemmas presented above follow from [9, Theorem 3.1], and [9, Theorem 3.7(3)] respectively. To see it one has to modify the proof of Lemma 10 to obtain that every continuous map with the average shadowing and shadowing properties has dense small periodic sets. Since minimal points are dense for every continuous map with dense small periodic sets the results from [9] apply.

Lemma 12. *Let $f: X \mapsto X$ be a surjective map of a compact metric space. If f has the specification property and the shadowing property, then f has the average shadowing property.*

Proof. Without loss of generality we may assume that $\operatorname{diam} X < 1$. Fix any $\varepsilon > 0$. It follows from the specification property and compactness of the space X that there is an integer N such that for any sequence $\{y_n\}_{n=0}^{\infty} \subset X$ and any infinite sequences of integers $\{j_n\}_{n=0}^{\infty}$ and $\{k_n\}_{n=0}^{\infty}$ fulfilling

$$0 = j_0 \leqslant k_0 < j_1 \leqslant k_1 < j_2 \leqslant k_2 < \cdots$$

and $j_{n+1} - k_n \geqslant N$ for $n = 0, 1, 2, \ldots$, there is a point $x \in X$ such that, for all integers i with $j_n \leqslant i \leqslant k_n$ for some $n \geqslant 0$ we have $d(f^i(x), f^i(y_n)) < \varepsilon/4$.

Take M such that $2N/M < \varepsilon/4$. Let $\delta > 0$ be provided by the shadowing property for $\varepsilon/4$ and put $\xi = \delta/M$.

Take any ξ-average-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$. We are going to show that there is a point which ε-traces it in average. First, we split $\{x_n\}_{n=0}^{\infty}$ into pieces which are maximal finite δ-pseudo-orbits. Strictly speaking, we put $i_0 = 0$ and if i_n is defined for some $n \geqslant 0$ then i_{n+1} is the largest integer such that $d(f(x_j), x_{j+1}) < \delta$ for all $i_n \leqslant j < i_{n+1}$. Without loss of generality we may assume that we get an infinite strictly increasing sequence $i_0 < i_1 < i_2 < \cdots$. For any $i_n \leqslant j < i_{n+1}$ we set $\phi(j) = n$, hence we obtain a function $\phi : \mathbb{N} \to \mathbb{N}$.

To make use of the specification property we need two infinite sequences of integers $\{j_n\}_{n=0}^{\infty}$ and $\{k_n\}_{n=0}^{\infty}$ fulfilling $0 = j_0 \leqslant k_0 < j_1 \leqslant k_1 < \cdots$ and $j_{n+1} - k_n = N$ for $n = 0, 1, 2, \ldots$. We define them inductively. We start with $j_0 = 0$. If j_n is defined for some n then we put $k_n = i_{\phi(j_n)+1} - 1$ and $j_{n+1} = k_n + N$.

By shadowing (and the fact that f is surjective), we can define a sequence of points $\{y_n\}_{n=0}^{\infty}$ such that for any $n \geqslant 1$ and $i_n \leqslant j < i_{n+1}$ we have $d(f^j(y_n), x_j) < \varepsilon/4$.

Then, by the above observation, there is a point $z \in X$ such that if $n \geqslant 0$ then $d(f^j(z), f^j(y_{\phi(j_n)})) < \varepsilon/4$ for any $j_n \leqslant j \leqslant k_n$. In particular, $d(f^j(z), x_j) < \varepsilon/2$ for n and j as above.

We claim that z ε-traces in average our ξ-average-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$. To prove our claim we fix any $l \geqslant 0$, large enough for

$$\frac{1}{n}\sum_{i=0}^{n-1} d\bigl(f(x_i), x_{i+1}\bigr) < \xi$$

to hold for all $n \geqslant l$. Let $l = sM + t$ for some $s \geqslant 0$, and $0 \leqslant t < M$. Then $\#\{i \leqslant n\colon\ d(f(x_i), x_{i+1}) \geqslant \delta\} < s+1$ since $(s+1)\delta/l \geqslant \delta/M = \xi$. Moreover, we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1} d\bigl(f^j(z), x_i\bigr) &\leqslant \frac{1}{n}\frac{\varepsilon}{2}\#\bigl\{0 \leqslant j < n\colon\ d\bigl(f^j(z), x_j\bigr) < \varepsilon/2\bigr\} + \frac{1}{n}\#\bigl\{0 \leqslant j < n\colon\ d\bigl(f^j(z), x_j\bigr) \geqslant \varepsilon/2\bigr\} \\
&\leqslant \frac{\varepsilon}{2} + \frac{N}{n}\#\bigl\{0 \leqslant j < n\colon\ d\bigl(f(x_j), x_{j+1}\bigr) \geqslant \delta\bigr\} \\
&\leqslant \frac{\varepsilon}{2} + \frac{N}{sM}(s+1) \leqslant \frac{\varepsilon}{2} + \frac{2N}{M} < \varepsilon,
\end{aligned}$$

and the proof is complete. □

Now, we are in position to prove our main theorem:

Theorem 1. *Let X be a compact metric space. If $f : X \to X$ is a continuous map with the shadowing property, then the following conditions are equivalent*:

(1) *f is totally transitive,*
(2) *f is topologically weakly mixing,*
(3) *f is topologically mixing,*
(4) *f is surjective and has the specification property,*
(5) *f is surjective and has the average shadowing property.*

Moreover, if f is in addition a c-expansive map, then any of the above conditions is equivalent to the periodic specification property of f.

Proof. By Lemma 6 and Lemma 7 we get that (1) implies (2). We use Lemma 8, Lemma 9, and Lemma 12, respectively, to prove that (2) implies (3), (3) implies (4), and (4) implies (5). Next, we observe that the implication from (4) to (1) is trivial, since it is immediate to check that every surjective map with the specification property is totally transitive (the assumption that f has shadowing property is not needed for this implication). Therefore conditions from (1) to (4) are equivalent under the assumption of shadowing of f. To finish the proof we apply Lemma 11 to see that (5) implies (2). □

5. Backward orbits for positively expansive maps

In this section we prove the corrected form of [13, Lemma 3]. In the statement presented in [13] f is a positively expansive open map, but in addition only transitivity of a map f is assumed, while total transitivity (or, equivalently, weak mixing) is necessary. It can easily be verified that the system presented in Example 3 is a valid counterexample.

It was proved long time ago by Reddy (see [1, Theorem 2.2.10]) that positively expansive maps are expanding. We will need that result in the following form.

Theorem 13 *(Reddy). If X is a compact metric space and $f: X \to X$ is a positively expansive continuous map, then there exists a metric ρ compatible with the topology of X such that f expands small ρ-distances, that is, one can find constants $\lambda > 1$ and $\delta > 0$ such that for all $x, y \in X$ we have*

$$\rho(x, y) \leqslant \delta \quad \Longrightarrow \quad \lambda\rho(x, y) \leqslant \rho\big(f(x), f(y)\big).$$

If we assume additionally that f is an open map then for some $0 < \eta < \delta/2$ if $\rho(f(p), q) < \lambda\eta$ for some $p, q \in X$, then there exits unique point $r \in X$ such that $\rho(p, r) < \eta$ and $f(r) = q$.

Recall that a sequence $\{x_{-n}\}_{n=0}^{\infty} \subset X$ is a *backward orbit* of f if and only if $f(x_{-n}) = x_{-n+1}$ for all $n > 0$. Let X_f denote the set of all backward orbits for f. The following theorem is the modified version of [13, Lemma 3]. Note that the assumption of weak mixing of f is equivalent to the assumption of total transitivity of f, since every positively expansive open map has the shadowing property by [1, Theorem 2.3.10], hence we may use our Theorem 1.

Theorem 14. *Let f be a continuous positively expansive, open and weakly mixing map acting on a compact metric space (X, d). Then there is a compatible metric ρ on X and there are constants $\beta > 0$, $\lambda > 1$ such that for any $(\{x_{-n}\}_{n=0}^{\infty}, y) \in X_f \times X$ there is a backward orbit $\{z_{-n}\}_{n=0}^{\infty}$ with*

(1) $z_0 = y$,
(2) $\rho(x_{-n}, z_{-n}) \leqslant \beta\lambda^{-n}\rho(x_0, y)$.

for all $n \geqslant 0$.

Proof. Let the constants $\eta > 0$, $\lambda > 1$, and the metric ρ be provided by Theorem 13. Take an integer N such that $\lambda^{-N} < \eta/2$. Let $V_1, \dots, V_M$ be an open cover of X such that every V_i has the diameter smaller than $\eta/2$. By weak mixing of f, there is $K \geqslant N$ such that for any $1 \leqslant i, j \leqslant M$ there is a point $q_{i,j} \in V_i$ with $f^K(q_{i,j}) \in V_j$. Put $\beta = \lambda^N \max\{\operatorname{diam} X \cdot \eta^{-1}, 1\}$.

By Theorem 13, for any $p, q \in X$ such that $\rho(p, f(q)) < \eta$ there is a point $r \in X$ such that $f(r) = p$ and $\lambda\rho(r, q) \leqslant \rho(f(r), f(q))$. Let s, t be such that $y \in V_t$ and $x_{-K} \in V_s$. Then $\rho(x_{-K}, q_{s,t}) < \eta/2$ and $\rho(y, f^K(q_{s,t})) < \eta/2$.

Now, we have to consider two cases. In the first case $\rho(y, x_0) < \eta$. Then we set $x'_{-k} = x_{-k}$ for $k = 0, \dots, K$. In the second case, $\rho(y, x_0) \geqslant \eta$, we set $x'_{-k} = f^{K-k}(q_{s,t})$ for $k = 0, \dots, K$.

In any case, $\rho(y, x'_0) = \rho(y, f(x'_{-1})) < \eta$, so Theorem 13 implies that there is z_{-1} such that $\rho(z_{-1}, x'_{-1}) < \eta$ and $f(z_{-1}) = z_0 = y$. Hence, $\rho(z_0, x'_0) \geqslant \lambda\rho(z_{-1}, x'_{-1})$. We can continue this procedure inductively, obtaining a backward orbit $z_0, z_{-1}, z_{-2}, \dots, z_{-K}$ such that $\rho(z_{-k+1}, x'_{-k+1}) \geqslant \lambda\rho(z_{-k}, x'_{-k})$ for every $k = 1, \dots, K$. In any case we have

$$\rho(z_{-j}, x_{-j}) \leqslant \operatorname{diam} X \leqslant \lambda^{-j}\beta\eta \leqslant \beta\lambda^{-j}\rho(x_0, y)$$

for $0 \leqslant j \leqslant K$. Since

$$\rho(z_{-K}, x_{-K}) \leqslant \rho(x_{-K}, q_{s,t}) + \rho(q_{s,t}, z_{-K}) < \eta,$$

Theorem 13 implies that there is z_{-K-1} such that $\rho(z_{-K-1}, x_{-K-1}) < \eta$ and $f(z_{-K-1}) = z_{-K}$. By induction we can continue our construction of a backward orbit obtaining $z_{-K-1}, z_{-K-2}, \dots$ with

$$\rho(z_{-K}, x_{-K}) \geqslant \lambda\rho(z_{-K-1}, x_{-K-1}) \geqslant \cdots \geqslant \lambda^j\rho(z_{-K-j}, x_{-K-j}),$$

for every $j \geqslant 1$. This completes the proof. □

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